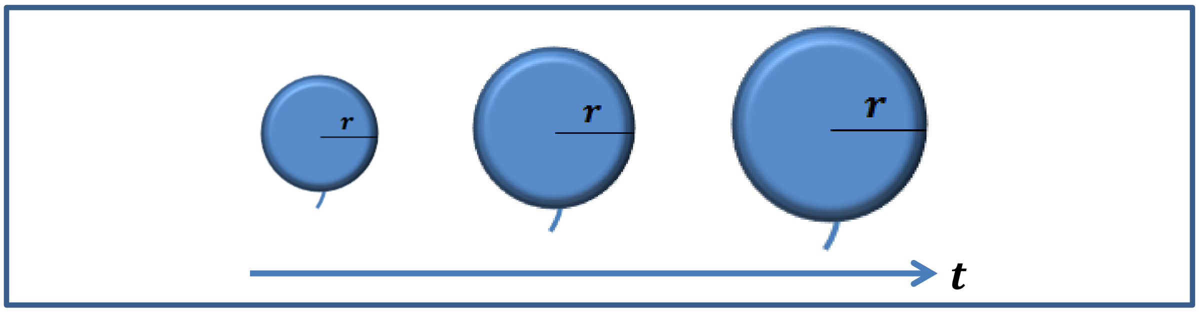
Related Rates

Related rate problems are an application of implicit differentiation. Here are some real-life examples to illustrate its use.

Example 1: Jamie is pumping air into a spherical balloon at a rate of $6\ \text{cm}^3/\text{min}$. What is the rate of change of the radius when the balloon has a radius of 12 cm?

How does implicit differentiation apply to this problem?

We must first understand that as a balloon gets filled with air, its radius and volume become larger and larger. As a result, its volume and radius are *related* to time. Hence, the term *related rates*.



In the question, it's stated that air is being pumped at a **rate of $\mathbf{6\ cm^3/min}$**. The key word being, **rate**. Since rate implies differentiation, we are actually looking at the change in volume over time.

Thus, $$6\ \text{cm}^3/\ \text{min} = v' = \frac{dV}{dt}.$$

In the second sentence, we are asked to find the **rate of change of the radius**. Again, rate implies a change in radius over time.

Thus, $r' = \frac{dr}{dt}$ = rate of change of the radius over time

Now that we understand what the question tells us, our objective is to find an equation that relates all of our given information.

In this case, the equation is the **volume of a sphere**: $$V = \frac{4}{3}\pi r^3.$$

Solution: Below is a chart summarizing our known and unknown values.

Known	**Unknown**
• $v' = \frac{dV}{dt} = 6\ \text{cm}^3/\text{min}$	• $r' = \frac{dr}{dt}$ - rate of change of the radius over time when r = 12 cm?

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Step 1: Given the equation for the volume of a sphere, multiply each term by $\frac{\boldsymbol{d}}{\boldsymbol{dt}}$.	$V = \frac{4}{3}\pi r^3$ $\frac{\boldsymbol{d}}{\boldsymbol{dt}} V = \frac{4}{3}\pi r^3 \frac{\boldsymbol{d}}{\boldsymbol{dt}}$
Step 2: Apply implicit differentiation. 1. Derive $r^3$with respect to r and 2. Multiply the result by $\frac{\boldsymbol{dr}}{\boldsymbol{dt}}$.	$\frac{\boldsymbol{d}V}{\boldsymbol{dt}} = 4\pi r^2 \frac{\boldsymbol{dr}}{\boldsymbol{dt}}$
Step 3: Substitute our known values into the equation. $\frac{dV}{dt} = 6 \text{ cm}^3/\text{min}$ and r =12 cm	$\boldsymbol{6} = 4\pi(\boldsymbol{12})^2 \frac{dr}{dt}$
Step 4: Simplify.	$6 = 576\pi \frac{dr}{dt}$
Step 5: Solve for $\frac{dr}{dt}$.	$\frac{dr}{dt} = \frac{1}{96\pi} cm/min$

Therefore, the change in radius over time when r = 12 cm is $\frac{1}{96\pi} cm/min$.

Example 2: James is filling an ice cream cone. The cone is 12 cm tall and has a radius of 4 cm. If the ice cream fills the cone evenly at a rate of $1.5 \text{ cm}^3/s$, what is the rate of change of the height when the height is 5 cm?

Solution: Below is a chart summarizing our known and unknown values.

Known	Unknown
• Cone is 12 cm tall with a 4 cm radius • $v' = \frac{dV}{dt} = \ 1.5 \text{ cm}^3/s$	• $h' = \frac{dh}{dt}$ – the rate of change of the height over time when h = 5 cm?

Understanding what our known and unknown values tell us about the situation, our objective becomes finding an equation that relates our given information.

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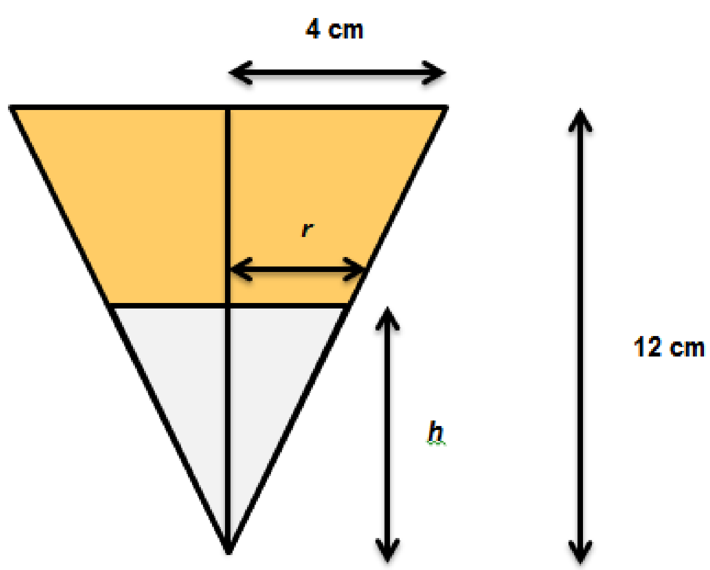
In this case, the equation is the **volume of a cone**: $V = \frac{1}{3}\pi r^2 h$.

Looking back at Example 1, our first step is to take the derivative of the volume equation with respect to time.

Step 1: Given the equation for the volume of a cone, multiply each term by $\frac{d}{dt}$.	$V = \frac{1}{3}\pi r^2 h$ $\frac{d}{dt}V = \frac{1}{3}\pi r^2 h \frac{d}{dt}$
Step 2: Apply the product rule between r^2 and h while following the rules for implicit differentiation.	$\frac{dV}{dt} = \frac{1}{3}\pi(r^2 h \frac{d}{dt})$ $\frac{dV}{dt} = \frac{\pi}{3}(2rh\frac{dr}{dt} + r^2\frac{dh}{dt})$ $\frac{dV}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt} + \frac{1}{3}\pi r^2\frac{dh}{dt}$
NOTE: $\frac{dr}{dt}$ or the value of r when $h = 5$ is not provided nor is it easily obtainable. We must analyze the cone further in order to find an alternative solution.	

When faced with related rate problems, it is sometimes helpful to sketch our problem. Here is the ice cream cone viewed from the side



Since our ice cream cone contains similar triangles we can use the ratio, $\frac{r}{h} = \frac{4}{12}$ to compare the cones radius and height.

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Solving for **r** we have: $\mathbf{r = \frac{1}{3}h}$

Given $\mathbf{r} = \frac{\mathbf{1}}{\mathbf{3}}\mathbf{h}$, we can make the following substitution in our volume equation:

$$V = \frac{1}{3}\pi \mathbf{r^2} h$$

$$V = \frac{1}{3}\pi \left(\frac{\mathbf{1}}{\mathbf{3}}\mathbf{h}\right)^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{\mathbf{1}}{\mathbf{9}}\boldsymbol{h^2}\right) h$$

$$V = \frac{\pi}{27}h^3$$

Now that our volume equation has been simplified, let's try taking the derivative with respect to time.

Step 1: Given the simplified equation for the volume of a cone, multiply each term by $\boldsymbol{\frac{d}{dt}}$.	$\boldsymbol{\frac{d}{dt}}V = \frac{\pi}{27}h^3 \boldsymbol{\frac{d}{dt}}$
Step 2: Apply implicit differentiation. 1. Derive $h^3$with respect to h and 2. Multiply the result by $\boldsymbol{\frac{dh}{dt}}$.	$\frac{dV}{dt} = \frac{\pi}{9}h^2 \frac{dh}{dt}$
Step 3: Substitute our known values into the equation.	$1.5 = \frac{\pi}{9}(5)^2 \frac{dh}{dt}$
Step 4: Simplify.	$\frac{3}{2} = \frac{25\pi}{9}\frac{dh}{dt}$
Step 5: Solve for $\frac{dh}{dt}$.	$\frac{dh}{dt} = \frac{27}{50\pi}$ cm/s

Therefore, the change in height over time is $\frac{27}{50\pi}\frac{\text{cm}}{\text{s}}$ when $\text{h} = 5$ cm.

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Exercises:

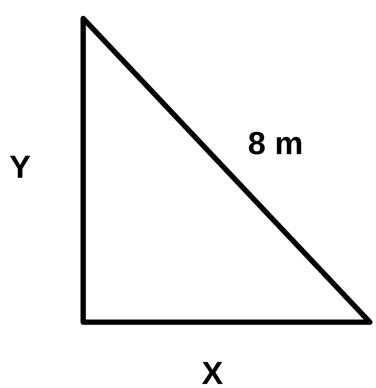
1. A ladder 8 m long leans against a model home. If the bottom of the ladder slides away from the building horizontally at a rate of 0.75 m/sec, how fast is the ladder sliding down the model home when the top of the ladder is 5 m from the ground?

2. You are looking at the New York ball drop on New Year's Eve at a distance of 100 m away from the base of the structure. If the ball drops at a constant rate of 2 m/s, what is the rate of change of the angle between you and the ball when the angle is $\pi/4$?

3. A car and a plane are racing each other. Both start at the same position, with the plane being 500 m above the car. The car has a maximum speed of 200 km/h and the plane has a maximum speed of 300 km/h. What is the rate of change of the distance between the plane and the car when both are at maximum speed and the plane is ahead by 70 m?

4. A cylindrical fountain is filled with juice. The can has a 10 cm radius. How fast does the height of the juice in the can drop when the drink is being drained at 5 cm^3/sec?

Solutions:

1. 	$x^2 + y^2 = 8^2$ $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ $2(6.24)(0.75)$ $+ 2(5)\frac{dy}{dt} = 0$ $9.36 + 10\frac{dy}{dt} = 0$ $\frac{dy}{dt} = -0.936\ m/sec$	Notes: $x^2 + y^2 = 8^2$ $x^2 + (5)^2 = 8^2$ $x = 6.24\ m$

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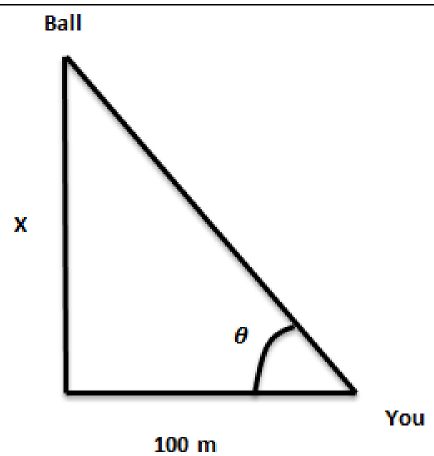
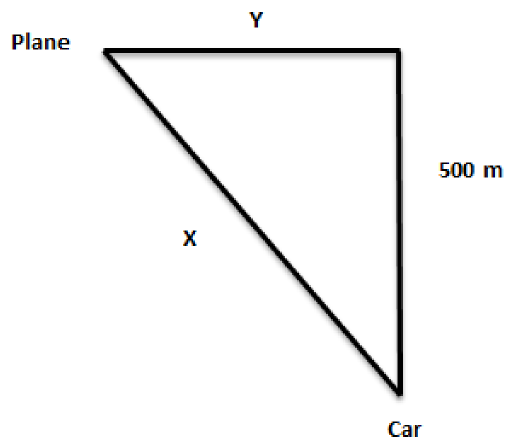
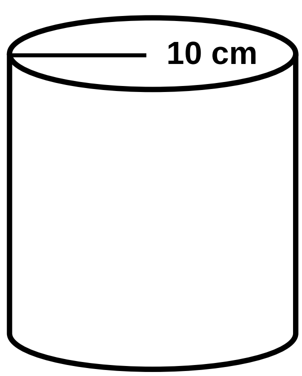
2. 	$\frac{x}{100} = \tan\theta$ $\frac{1}{100}\frac{dx}{dt} = \sec^2(\theta)\frac{d\theta}{dt}$ $\frac{1}{100}(2) = (2)\frac{d\theta}{dt}$ $\frac{d\theta}{dt} = \frac{1}{100} rad/s$	
3. 	$y^2 + 500^2 = x^2$ $2y\frac{dy}{dt} + 500^2 = 2x\frac{dx}{dt}$ $2(70)(100) =$ $2(504.88)\frac{dx}{dt}$ $(14000) =$ $(1009.76)\frac{dx}{dt}$ 13.86 km/h= $\frac{dx}{dt}$	Notes: $\frac{dy}{dt} = 100\frac{km}{h} =$ difference between two speeds $y^2 + 500^2 = x^2$ $(70)^2 + 500^2 = x^2$ $x \cong 504.88$
4. 	$V = \pi r^2 h$ $\frac{dV}{dt} = \pi(2rh\frac{dr}{dt} + \frac{dh}{dt}r^2)$ $-5 = \pi(2(10)(0) + \frac{dh}{dt}(10)^2)$ $-5 = \pi(100\frac{dh}{dt})$ $\frac{-1}{20} = \frac{dh}{dt}$	