Radicals

A radical expression is an expression involving the root symbol ($\sqrt{\ }$). The root symbol itself, is called the *radical*. The *radicand*, represented by the value inside the root symbol is the number that will be operated on, and the *index* of the root represented by the value outside the root describes the type of operation:

index → $\sqrt[4]{81}$ ← **radical** ← **radicand**

In general, radical expressions are of the form: $\sqrt[n]{x^m}$

Roots and Exponents

Roots and exponents are related.

An *exponential expression* with a fractional exponent can be expressed as a *radical* where the denominator is the index of the root, and the numerator remains as the exponent.

$\boldsymbol{a^{\frac{m}{n}}}$ ⇔ $\boldsymbol{\sqrt[n]{a^m} = (\sqrt[n]{a})^m}$

exponent **radical**

Example 1: Write $125^{\frac{1}{3}}$ as a radical expression.

$$125^{\frac{1}{3}} = \sqrt[3]{125^1} = \sqrt[3]{125}$$

Finding Roots

In math, every operation has an opposite operation (for example, multiplication/division and addition/subtraction). The root operation is the opposite of the exponent operation.

Example 2: Find the square root of x^2 (i.e $\sqrt[2]{\boldsymbol{x^2}}$).

Note – The *index* of a *square root* is two (2). Since square roots are so commonly used it's typical for the index number to not be written. $\sqrt{\ } = \sqrt[2]{\ }$

$$\sqrt[2]{x^2} = x^{2/2} = x^1 = x$$

Radicals

To solve a radical expression we can break the radicand into its prime factors. If the radicand can be written as an exponent raised to a number equal to the index, then the exponent will cancel out.

Example 3: $\sqrt[5]{32}$

$= 32^{\frac{1}{5}}$
$= (16\ x\ 2)^{\frac{1}{5}}$
$= (4\ x\ 4\ x\ 2)^{\frac{1}{5}}$
$= (2\ x\ 2\ x\ 2\ x\ 2\ x\ 2)^{\frac{1}{5}}$
$= (2^5)^{\frac{1}{5}}$
$= 2^{5/5}$
$= 2$

Example 4: $\sqrt[6]{729}$

$= (729)^{\frac{1}{6}}$
$= (243\ x\ 3)^{\frac{1}{6}}$
$= (81\ x\ 3\ x\ 3)^{\frac{1}{6}}$
$= (3\ x\ 3\ \ x\ 3\ x\ 3\ x\ 3\ x\ 3)^{\frac{1}{6}}$
$= (3^6)^{\frac{1}{6}}$
$= 3^{6/6}$
$= 3$

If the radicand cannot be broken down into a prime factor raised to an exponent equal in number to the index, then the following Radical Rules can be applied.

Radical Laws	Examples
1. $\sqrt[n]{ab} = \sqrt[n]{a}\ \sqrt[n]{b}$	$\sqrt[3]{(-8)(27)} = \sqrt[3]{-8}\ \sqrt[3]{27} = (-2)(3) = -6$
2. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt[4]{\frac{16}{81}} = \frac{\sqrt[4]{16}}{\sqrt[4]{81}} = 2/3$
3. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt{\sqrt[3]{729}} = \sqrt[6]{729} = 3$
4. $\sqrt[n]{a^n} = a$**, if n is an odd number**	$\sqrt[3]{(-5)^3} = -5$

5. $\sqrt[n]{a^n} = \lvert a \rvert$**, if n is an even number**	$\sqrt[4]{(-3)^4} = \lvert -3 \rvert = 3$

Note: $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$

$\sqrt[n]{a-b} \neq \sqrt[n]{a} - \sqrt[n]{b}$

$\sqrt[n]{a^n+b^n} \neq a+b$

Example 5:

$\sqrt[4]{81x^8y^4}$

$= \sqrt[4]{81}\ \sqrt[4]{x^8}\ \sqrt[4]{y^4}$

$= 81^{\frac{1}{4}}\ (x^{\frac{8}{4}})\ (y^{\frac{4}{4}})$

$= (3^4)^{\frac{1}{4}}\ (x^2)(y)$

$= 3x^2y$

Example 6:

$\sqrt{32} + \sqrt{200}$

$= \sqrt{16(2)} + \sqrt{100(2)}$

$= \sqrt{16}\,\sqrt{2} + \sqrt{100}\,\sqrt{2}$

$= 16^{\frac{1}{2}}\sqrt{2} + 100^{\frac{1}{2}}\sqrt{2}$

$= (4^2)^{\frac{1}{2}}\sqrt{2} + (10^2)^{\frac{1}{2}}\sqrt{2}$

$= 4\sqrt{2} + 10\sqrt{2}$

$= \sqrt{2}\,(4+10)$

$= 14\sqrt{2}$

Radicals

Exercises:

1. Express the following exponents as radical expressions.

   a) $4^{\frac{2}{3}} =$

   b) $25^{\frac{1}{2}} =$

   c) $3^{\frac{4}{5}} =$

2. Express the following radicals as exponential expressions.

   a) $\sqrt{81} =$
   b) $\sqrt[3]{64} =$
   c) $\sqrt[5]{243^3} =$

3. Find the square root of the following numbers.

   a) $x^2 = 49$
   b) $x^2 = 144$

4. Simplify the following radical expressions.

   a) $\sqrt{81m^{64}} =$

   b) $\sqrt{49a^4b^{12}} =$

   c) $\sqrt[3]{\frac{9x^6}{27}} =$

   d) $8\sqrt[3]{5} - 3\sqrt[3]{5} =$

   e) $\sqrt[3]{54} + \sqrt[3]{128} =$

   f) $\frac{\sqrt{x^4}}{\sqrt{y^5}} =$

Radicals

Solutions:

1. Express the following exponents as radical expressions.

   a) $4^{\frac{2}{3}} = \boldsymbol{\sqrt[3]{4^2}}$
   b) $25^{\frac{1}{2}} = \boldsymbol{\sqrt{25}}$
   c) $3^{\frac{4}{5}} = \boldsymbol{\sqrt[5]{3^4}}$

2. Express the following radicals as exponential expressions.

   a) $\sqrt{81} = \boldsymbol{81^{\frac{1}{2}}}$
   b) $\sqrt[3]{64} = \boldsymbol{64^{\frac{1}{3}}}$
   c) $\sqrt[5]{243^3} = \boldsymbol{243^{\frac{3}{5}}}$

3. Find the square root of the following numbers.

   a) $x^2 = 49,\ \boldsymbol{x = \pm 7}$
   b) $x^2 = 144,\ \boldsymbol{x = \pm 12}$

4. Simplify the following radical expressions.

   a) $\sqrt{81m^{64}} = \boldsymbol{9m^{32}}$

   b) $\sqrt{49a^4b^{12}} = \boldsymbol{7a^2b^6}$

   c) $\sqrt[3]{\frac{9x^6}{27}} = \boldsymbol{\frac{\sqrt[3]{9}\,x^2}{3}}$

   d) $8\sqrt[3]{5} - 3\sqrt[3]{5} = \boldsymbol{5\sqrt[3]{5}}$ **or** $\boldsymbol{\sqrt[3]{625}}$

   e) $\sqrt[3]{54} + \sqrt[3]{128} = \boldsymbol{7\sqrt[3]{2}}$

   f) $\frac{\sqrt{x^4}}{\sqrt{y^5}} = x^2y^{\frac{-5}{2}}$ **or** $\boldsymbol{\frac{x^2}{y^2\sqrt{y}}}$