Factoring Quadratic Expressions: Special Cases

Special case 1: Difference of Squares

Difference of squares is a special case of factoring, which follows a specific pattern. Firstly, it is important to be able to recognize a difference of squares.

> For an algebraic expression to be a difference of squares **the first and last terms must be perfect squares. The two perfect squares must be subtracted.**

Perfect square – a number whose square root is a whole number and/or a variable with an even exponent.

Numbers that are perfect squares: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, …

Variables are perfect squares if they have an ***even*** exponent (e.g. 2, 4, 6, 8, etc.)

Examples of Difference of Squares			NOT Examples of Difference of Squares		
$4x^2 - 9$	$16y^4 - 25$	$169p^8 - 1$	$4x^2 \boxed{+} 9$	$16y^{\boxed{5}} - 25$	$\boxed{168}p^8 - 1$

If you can recognize a difference of squares, the factoring can be done in one line according to the pattern below.

> **Factoring a difference of squares:**
>
> $$a^2 - c^2 = (a + c)\,(a - c)$$

Notice that the first term inside both brackets is the square root of a^2; the second term inside both brackets is the square root of c^2; the operation sign inside one bracket is + and inside the other bracket is −.

Example:

Factor. $25x^2 - 121$.

Step 1: Try to common factor first. There are no common factors of $25x^2$ and 121.

Step 2: Recognize that this is a difference of squares since $25x^2$ and 121 are both perfect squares AND these two perfect squares are being subtracted.

Step 3: Factor according to $a^2 - c^2 = (a + c)\,(a - c)$.

$$25x^2 - 121 = (\sqrt{25x^2} + \sqrt{121})\,(\sqrt{25x^2} - \sqrt{121})$$

$$= (5x + 11)(5x - 11)$$

Step 4 (Optional): To double check the answer, expand $(5x + 11)(5x – 11)$. The result of expanding should be $25x^2 – 121$.

$(5x + 11)(5x – 11)$

$= 25x^2 – 55x + 55x – 121$

$= 25x^2 – 121 + 0x$

$= 25x^2 – 121$

A difference of squares can sometimes be "disguised" when it is multiplied by a common factor. This is why it's important to **always try to common factor an algebraic expression first**.

Example:

Factor. $98x^3 – 2x$.

Step 1: Common factor the expression first.

98 and 2 are both divisible by 2

x^3 and x are both divisible by x

Thus, the common factor of $98x^3$ and $2x$ is $2x$.

$98x^3 – 2x = 2x(49x^2 – 1)$

Notice that the algebraic expression inside the brackets, $49x^2 – 1$, is a difference of squares.

Step 2: Factor the difference of squares according to the pattern.

$49x^2 – 1 = (\sqrt{49x^2} + \sqrt{1})\ (\sqrt{49x^2} - \sqrt{1})$

$= (7x + 1)(7x – 1)$

Step 3: Write the final answer by putting all of the factors together.

$98x^3 – 2x = 2x(7x + 1)(7x – 1)$

Step 4 (Optional): To double check the answer, expand $2x(7x + 1)(7x – 1)$. The result of expanding should be $98x^3 – 2x$.

$2x(7x + 1)(7x – 1)$

Multiply 2x by each term in the first bracket, 7x + 1. (Note: 2x does NOT get multiplied by *each* bracket since you would be multiplying 2x twice).

$= (14x^2 + 2x)(7x - 1)$	Use FOIL method to multiply binomial by a binomial.
$= 98x^3 - 2x + 14x^2 - 14x^2$	Collect like terms.
$= 98x^3 - 2x + 0x^2$	Simplify $0x^2$.
$= 98x^3 - 2x$	

Special case 2: Perfect Square Trinomial

Perfect square trinomial is another special case of factoring, which follows a specific pattern. Firstly, it is important to be able to recognize a perfect square trinomial.

> For an algebraic expression to be a perfect square trinomial **the first and last terms must be perfect squares. The middle term has to equal to twice the square root of the first term times the square root of the last term.**

Examples of Perfect Square Trinomials	NOT Examples of Perfect Square Trinomials	
$x^2 + 12x + 36$	$x^2 + 12x \boxed{-36}$	-36 is NOT a perfect square
$16y^2 - 40y + 25$	$16y^2 \boxed{-41}y + 25$	$-41y$ does NOT equal to $2(\sqrt{16y^2})(\sqrt{25})$
$169p^2 + 26p - 1$	$\boxed{168p^2} + 26p - 1$	$168p^2$ is NOT a perfect square
$4x^2 + 12x + 9$	$4x^2 + 12x \boxed{+8}$	8 is NOT a perfect square

If you can recognize a perfect square trinomial, the factoring can be done in one line according to the pattern below.

Factoring a perfect square trinomial:

$$a^2 + 2ac + c^2 = (a + c)(a + c) = (a + c)^2$$

$$a^2 - 2ac + c^2 = (a - c)(a - c) = (a - c)^2$$

Notice that the first term inside both brackets is the square root of a^2; the second term inside both brackets is the square root of c^2; the operation sign (+ or −) inside the brackets is the same as the operation sign in front of 2ac.

Example:

Factor. $36x^2 - 132x + 121$.

<u>Step 1:</u> Try to common factor first. There are no common factors of $36x^2$, 132x and 121.

<u>Step 2:</u> Recognize that this is a perfect square trinomial since $36x^2$ and 121 are both perfect squares AND the middle term, 132x, is equal to $2(\sqrt{36x^2})(\sqrt{121}) = 2(6x)(11)$

<u>Step 3:</u> Factor according to $a^2 - 2ac + c^2 = (a - c)^2$

$$36x^2 - 132x + 121 = (\sqrt{36x^2} - \sqrt{121})^2 = (6x - 11)^2$$

$$a^2 \quad - 2ac \quad + \quad c^2 \quad = \quad (\sqrt{a^2} - \sqrt{c^2})^2 \quad = (a - c)^2$$

<u>Step 4 (Optional):</u> To double check the answer, expand $(6x - 11)^2$. The result of expanding should be $36x^2 - 132x + 121$.

$$(6x - 11)^2 = (6x - 11)(6x - 11)$$

$= 36x^2 - 66x - 66x + 121$

$= 36x^2 - 132x + 121$

A perfect square trinomial can sometimes be "disguised" when it is multiplied by a common factor. This is why it's important to **always try to common factor an algebraic expression first**.

Example:

Factor. $-18x^3 - 96x^2 - 128x$

<u>Step 1:</u> Common factor the expression first.

$-18x^3$, $-96x^2$ and $-128x$ are all divisible by $-2x$.

Thus, the common factor is $-2x$.

$-18x^3 - 96x^2 - 128x = -2x(9x^2 + 48x + 64)$

Notice that the algebraic expression inside the brackets, $9x^2 + 48x + 64$, is a perfect square trinomial.

Step 2: Factor the perfect square trinomial according to $a^2 + 2ac + c^2 = (a + c)^2$

$$9x^2 + 48x + 64 = (\sqrt{9x^2} + \sqrt{64})^2 = (3x + 8)^2$$

$$a^2 + 2ac + c^2 = (\sqrt{a^2} + \sqrt{c^2})^2 = (a + c)^2$$

Step 3: Write the final answer by putting all the factors together.

$-18x^3 - 96x^2 - 128x = -2x(3x + 8)^2$

From Step 1. From Step 2.

Step 4 (Optional): To double check the answer, expand $-2x(3x + 8)^2$. The result of expanding should be $-18x^3 - 96x^2 - 128x$.

$-2x(3x + 8)^2 = -2x(3x + 8)(3x + 8)$

$= (-6x^2 - 16x)(3x + 8)$

$= -18x^3 - 48x^2 - 48x^2 - 128x$

$= -18x^3 - 96x^2 - 128x$

Practice Questions:

1. Determine if each of the following expressions is a difference of squares, a perfect square trinomial or neither.

a) $81q^3 - 16$
b) $121d^4 - d^2$
c) $9x^2 + 6xy + y^2$
d) $36x^2 + 40x + 25$

2. Factor.

a) $144y^2 - 49$

Answers:

1. a) neither
b) difference of squares
c) perfect square trinomial
d) neither

2. a) $(12y - 7)(12y + 7)$
b) $(8x^2 - 10x)(8x^2 + 10x)$
c) $(13y - 7)^2$
d) $3(y + 8)^2$
e) $2(y^2 - 1)(y^2 + 1)$

b) $64x^4 - 100x^2$
c) $169y^2 - 182y + 49$
d) $192 + 3y^2 + 48y$
e) $2y^4 - 2$